Indag. Mathem., N.S. 3 (4), 365-375 December 21, 1992

Plurisubharmonic functions and Kählerian metrics on complexification of symmetric spaces

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Communicated by Prof. T.A. Springer at the meeting of March 30, 1992

INTRODUCTION

This paper is in two parts:

I) Let $X=G/K$ be a compact symmetric space and $X_{\mathbb{C}}=G_{\mathbb{C}}/K_{\mathbb{C}}$ its complexification. Theorem 1 gives a characterisation of G-invariant plurisubharmonic functions on G-invariant Stein domains in $X_{\mathbb{C}}$ in terms of convexity. This generalises the classical case when G is abelian. This study was inspired by the work of M. Lasalle ([La1] and [La2]) and permits us also to find his results on G-invariant domains by different methods. In the proof of our results we use finite dimensional representation theory of complex reductive groups.

II) We use the results of I to give a characterization of G-invariant Kählerian metrics on $X_{\mathbb{C}}$ (or on invariant Stein domains therein). When K is semi-simple we have: $\omega=i\partial\bar{\partial}\varphi$ where φ is G-invariant and plurisubharmonic (theorem 2). This gives us, using theorem 1, a characterization of such ω in terms of convexity.

The appendix uses the method of II) to give a characterization of G-invariant Kählerian metrics on rational homogeneous manifolds $G_{\mathbb{C}}/P$. These results were found by another method in [Al-Pe].

I. COMPLEXIFICATION OF SYMMETRIC SPACES AND PLURISUBHARMONIC FUNCTIONS

Let D be a Reinhardt domain in $(\mathbb{C}^*)^n$ and denote by $\mathscr{D}$ its basis: $\mathscr{D}=\{(x_1,\dots,x_n)\in\mathbb{R}^n:(e^{x_1},\dots,e^{x_n})\in D\}$. The following results are classical:

R1: *D is a domain of holomorphy if and only if $\mathscr{D}$ is convex.*

R2: *A function f on D which is $(S^1)^n$ invariant is plurisubharmonic if and only if the function $\tilde{f}$ defined on $\mathscr{D}$ by $\tilde{f}(x_1, \dots, x_n) = f(e^{x_1}, \dots, e^{x_n})$ is convex.*

The result (R1) was extended by M. Lasalle to domains in a complex reductive Lie group which are biinvariant under the action of a maximal compact group G ([La1]) and then generalized for G-invariant domains of the complexification of compact symmetric spaces ([La2]). The aim of this paper is to generalize (R2) in the same geometric setting. In fact (R1) is then a corollary of (R2).

NOTATION. (see [La2]).

A) Let G be a compact Lie group with a given involution θ. Let G_θ be the subgroup of θ-fixed elements in G and G_θ^0 the connected component of the identity in G_θ. We fix a compact subgroup K of G_θ containing G_θ^0. Let $G_{\mathbb{C}}$ be the reductive group which is the complexification of G and $K_{\mathbb{C}}$ the complexification of K in $G_{\mathbb{C}}$. We put $\boldsymbol{g} = \mathrm{Lie}(G)$, $\boldsymbol{g}_{\mathbb{C}} = \mathrm{Lie}(G_{\mathbb{C}})$, $\boldsymbol{k} = \mathrm{Lie}(K)$ and $\boldsymbol{k}_{\mathbb{C}} = \mathrm{Lie}(K_{\mathbb{C}})$. We extend θ to a holomorphic involution on $G_{\mathbb{C}}$. This new involution, as well as the corresponding involution on $\dot{G}_{\mathbb{C}}$ are also denoted by θ. Let $\boldsymbol{l}$ be the subspace of $\boldsymbol{g}$ defined by:

$$\boldsymbol{l} = \{X \in \boldsymbol{g} \colon \theta(X) = -X\}.$$

We fix a maximal abelian subalgebra $\boldsymbol{a}$ in $i\boldsymbol{l}$. The exponential map is a diffeomorphism of $\boldsymbol{a}$ onto the corresponding Lie group A. The inverse map is denoted by $\log_A$. Let $N_K(A)$ and $Z_K(A)$ be the normalizer and centralizer of A in K. We put $N_K(A)/_{Z_K(A)} = W$. It is the Weyl group of the pair (G, K). It is a finite group and it operates on $\boldsymbol{a}$ by conjugation. Any element of $G_{\mathbb{C}}$ is of the form $g_0 \exp H \cdot k$ with $g_0 \in G$, $H \in \boldsymbol{a}$ and $k \in K_{\mathbb{C}}$ (see [La2]). This decomposition is not unique. The element H is defined up to conjugation under the Weyl group. Let $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$. It is a complexification of a compact symmetric space. We denote by p the canonical projection of $G_{\mathbb{C}}$ onto $X_{\mathbb{C}}$.

B) *Domains and functions*

To an open G-invariant set D in $X_{\mathbb{C}}$ one associates its basis $\mathscr{D}$ in A defined by: $\mathscr{D} = \log_A(p^{-1}D \cap A)$. It is a W-invariant open set in $\boldsymbol{a}$. The correspondence between G-invariant open sets in $G_{\mathbb{C}}$ and W-invariant open sets in $\boldsymbol{a}$ is bijective. For such a domain $\mathscr{D} = \log_A(p^{-1}D \cap A)$, we have the following sets of functions:

- $F_G(D)$ (resp. $C_G^\infty(D)$): G-invariant functions on D (resp. $C^\infty G$-invariant functions)
- $P_G(D)$: plurisubharmonic G-invariant functions on D
- $P_G^+(D)$: strictly plurisubharmonic G-invariant C^∞-functions on D
- $F_W(\mathscr{D})$: W-invariant functions on $\mathscr{D}$ and $C_W^\infty(\mathscr{D}) = F_W(\mathscr{D}) \cap C^\infty(\mathscr{D})$
- $\mathrm{Conv}_W(\mathscr{D})$: W-invariant convex functions on $\mathscr{D}$

- $\mathrm{Conv}_W^+(\mathscr{D})$: W-invariant C^∞-strictly convex functions on $\mathscr{D}$.

To a G-invariant function f on D we associate the function $\tilde{f}$ on $\mathscr{D}$, which is W-invariant and defined by: $\tilde{f}(H)=f(e^H)$. For the following theorem, see ([F.J], p. 118, th. 4.1).

THEOREM A. *The map $f\mapsto\tilde{f}$ is a bijection of $F_G(D)$ (resp. $C_G^\infty(D)$) onto $F_W(\mathscr{D})$ (resp. $C_W^\infty(\mathscr{D})$).*

REMARK. This theorem is proved in [F.J] for $D=G_{\mathbb{C}}$ reductive and in the C^∞-case. But the proof works easily in our situation.

Our theorem is now the following:

THEOREM 1. *For a G-invariant domain D in $X_{\mathbb{C}}$ the map $f\mapsto\tilde{f}$ is a bijection in the following cases:*

a) *$P_G(D)$ onto* $\mathrm{Conv}_W(\mathscr{D})$

b) *$P_G(D)\cap C^\infty(D)$ onto* $\mathrm{Conv}_W(\mathscr{D})\cap C^\infty(\mathscr{D})$

c) *$P_G^+(D)$ onto* $\mathrm{Conv}_W^+(\mathscr{D})$.

REMARKS ON THE THEOREM. 1) The theorem implies that f in $P_G(D)$ is continuous, because $\tilde{f}$ is convex and there is a theorem A for continuous functions.

2) Let H be a complex reductive group and M a maximal compact subgroup of G. Denote by $d(H)$ the diagonal in $H\times H$. Then H can be identified with $(H\times H)/_{d(H)}$ and the M-biinvariant functions on H can be identified with the $M\times M$ left-invariant functions on $(H\times H)/_{d(H)}$. But $H\times H/_{d(H)}$ is the complexification of the symmetric space $(M\times M)/_{d(M)}$. Then as a special case of the theorem we have a bijective correspondence between M-biinvariant plurisubharmonic functions and W-invariant functions on Lie(A) where A comes from the Cartan decomposition $G=KAK$ (see [La1] for the same theorem for domains).

For a G-invariant domain D in $X_{\mathbb{C}}$ with basis $\mathscr{D}$ one denotes by $\hat{D}$ the G-invariant domain in $X_{\mathbb{C}}$ with basis the convex hull of $\mathscr{D}$.

COROLLARY (result R1 of M. Lasalle). *$\hat{D}$ is the holomorphy hull of D.*

PROOF OF THE COROLLARY. Put $\Delta=p^{-1}D$. One knows ([Ro], see also [Co-Lo]) that every holomorphic function on Δ extends to $\hat{\Delta}$ which is the smallest left G-invariant domain in $G_{\mathbb{C}}$ containing Δ and such that the following property (P) is satisfied:

(P) $\forall g\in\hat{\Delta}$, $\forall X\in g$, $(\exp iX)g\in\hat{\Delta}$ implies

$$(\exp itX)\cdot g\in\hat{\Delta}\quad\forall t\in[0,1].$$

Notice that the image of $\hat{\Delta}$ in the Riemannian symmetric space $G\backslash G_{\mathbb{C}}$ is just the geodesic convex hull of the image of Δ in $G\backslash G_{\mathbb{C}}$. For $k\in K_{\mathbb{C}}$, the set $\hat{\Delta}\cdot k^{-1}$ is a G-invariant domain in $G_{\mathbb{C}}$ containing Δ, and such that (P) holds.

Hence $\hat{\Delta}\cdot k^{-1}\supset\hat{\Delta}$. It follows that $\hat{\Delta}\cdot k\subset\hat{\Delta}$, for every $k\in K_{\mathbb{C}}$. Hence $\hat{\Delta}$ is right $K_{\mathbb{C}}$-invariant, and the $K_{\mathbb{C}}$-invariant holomorphic functions on Δ extend to $K_{\mathbb{C}}$-invariant holomorphic functions on $\hat{\Delta}$. Now let $H, H'\in\log_A(\hat{\Delta}\cap A)$. Using property (P) with $g=\exp H$ and $X=(H'-H)/i$ we see that $\exp((1-t)H+tH')=(\exp itX)\cdot g\in\hat{\Delta}\cap A$ for all $t\in[0,1]$. Hence $\log_A(\hat{\Delta}\cap A)$ is convex in $\boldsymbol{a}$, so that it must contain the convex hull of $\mathscr{D}$. We conclude that $p^{-1}\hat{D}\subset\hat{\Delta}$, and therefore every holomorphic function in D extends holomorphically to $\hat{D}$.

It remains to prove that $\hat{D}$ is a domain of holomorphy. It suffices, by a theorem of Grauert [Hö] to show the existence on $\hat{D}$ of a regular, strictly plurisubharmonic exhaustion function f. This function is built as follows: Take on the basis $\hat{\mathscr{D}}$ of $\hat{D}$, which is convex, a regular strictly convex exhaustion function h. One can suppose (by taking a mean) that h is W-invariant. Then applying the theorem (part c) and the correspondence between K-invariant compacta in $X_{\mathbb{C}}$ and W-invariant compacta in $\boldsymbol{a}$, one concludes by taking f such that $\tilde{f}=h$.

The following is a little bit technical. It is a summary of results of [Ta] p. 448–457.

NOTATIONS AND PRELIMINARY REMARKS TO THEOREM 1 a) We choose on $\boldsymbol{g}_{\mathbb{C}}$ a complex symmetric bilinear form which is invariant under θ and $\mathrm{Ad}(G_{\mathbb{C}})$, and negative definite on $\boldsymbol{g}$. We denote this form by $(\ ,\ )$. One extends $i\boldsymbol{a}$ to a Cartan subalgebra $\boldsymbol{t}$ of $\boldsymbol{g}$ and one identifies $i\boldsymbol{t}$ (resp. $\boldsymbol{a}$) with its dual by the (negative) scalar product $(\ ,\ )$. Let $\boldsymbol{g}'=[\boldsymbol{g},\boldsymbol{g}]$ and $\boldsymbol{c}$ the center of $\boldsymbol{g}$. One has the decomposition: $\boldsymbol{g}=\boldsymbol{g}'\oplus\boldsymbol{c}$. For a subspace V of $\boldsymbol{g}$ put $V'=V\cap\boldsymbol{g}'$.

(b) *Roots*: Let Σ be a root system for $i\boldsymbol{t}$. One chooses on Σ an order $>$, with the property that $\alpha>0$, $\theta\alpha\neq\alpha\Rightarrow\theta\alpha<0$. Let $\{\alpha_1,\dots,\alpha_l\}$ be an ordering of the associated fundamental system such that $\alpha_i\neq\theta\alpha_i$, for $i=1,\dots,p$ and $\alpha_i=\theta\alpha_i$ for $i>p$, where p is the dimension of $\boldsymbol{a}$. Put $\gamma_i=\frac{1}{2}(\alpha_i-\theta\alpha_i)$ for $i\leq p$, and define $\pi=\{\beta_1,\dots,\beta_p\}$ in $\boldsymbol{a}$ with:

$$\beta_i=\begin{cases}\gamma_i & \text{if } 2\gamma_i \text{ is not a root}\\ 2\gamma_i & \text{if } 2\gamma_i \text{ is a root.}\end{cases}$$

For $\lambda\neq0$ in $\boldsymbol{a}$, put $\lambda^*=2\lambda/_{(\lambda,\lambda)}$. Denote by $\boldsymbol{a}^+$ the positive Weyl chamber in $\boldsymbol{a}$.

(c) Define the following sets:

$\Gamma=\{H\in i\boldsymbol{a}\colon \exp H\in K\}$. It is a lattice in $i\boldsymbol{a}$.

$\Gamma_c=\Gamma\cap\boldsymbol{c}$

$\Gamma_0'=\sum_{j=1}^{p}\mathbb{Z}(i\pi\beta_j^*)$

$\Gamma_0=\Gamma_c\oplus\Gamma_0'$ (It is a lattice in $i\boldsymbol{a}$)

$Z=\{\lambda\in\boldsymbol{a}\colon(\lambda,H)\in2\pi iZ,\quad\forall H\in\Gamma\}$

$\Delta=\{\lambda\in Z\colon(\lambda,\beta_i^*)\geq0,\quad\text{for } 1\leq i\leq p\}$.

If we choose a basis $\{2\pi i\delta_1,\dots,2\pi i\delta_r\}$ in Γ_c, then $\{\frac{1}{2}\beta_1^*,\dots,\frac{1}{2}\beta_p^*,\delta_1,\dots,\delta_r\}$ is a basis of $\boldsymbol{a}$. Let $B=\{\omega_1',\dots,\omega_p',\mu_1',\dots,\mu_r'\}$ be the dual basis. We have $B\subseteq\Delta$ and $(f,H_0)\in 2\pi i\mathbb{Z}$ if $f\in B$ and $H_0\in\Gamma_0$. As Γ_0 is a sublattice of Γ (Lemma 2 of [Ta], p. 451), this implies the existence of $n\in\mathbb{N}^*$ such that the elements $\omega_i=n\cdot\omega_i'$ and $\mu_i=n\mu_i'$ are in Z. They remain also in Δ. The link between Δ and representation theory is as follows: For every $\lambda\in\Delta$, there exists a holomorphic irreducible representation ϱ of class 1 (ϱ has a nonzero $K_{\mathbb{C}}$-invariant vector) with highest weight λ (see [Ta], 2.4, p. 457). If f is a function on $\boldsymbol{a}$ we denote by $\hat{f}$ the following W-invariant function: $\hat{f}(x)=\max_{w\in W}f(wx)$.

To prove the theorem, we need the following lemma:

LEMMA. *To every element L in $\boldsymbol{a}$ (identified with its dual) one can associate two sequences of real numbers $(t_i)_{1\le i\le p}$ and $(s_j)_{1\le j\le r}$ with $t_i\ge 0$ such that:*

$$\hat{L}=\sum_{i=1}^{p}t_i\omega_i+\sum_{j=1}^{r}s_j\mu_j.$$

PROOF. Using the W-action, one can suppose that $L\ge wL$ for every $w\in W$. The element L can be written uniquely as:

$$L=\sum_{i=1}^{p}t_i\omega_i+\sum_{j=1}^{r}s_j\mu_j.$$

We have $t_i=(L,\beta_i^*)$, which is positive by hypothesis. We also have $\hat{\mu}_j=\mu_j$ for every j as the μ_j come from the centre. The Lemma is then implied by the W-invariance of $\hat{L}$.

PROOF OF THE THEOREM (part *a*)

a) The injectivity of $f\mapsto\tilde{f}$ comes from theorem A.

b) We must prove that $f\in P_G(D)\Rightarrow\tilde{f}\in\mathrm{Conv}_W(\mathscr{D})$.

Let H_0 and H_0+H_1 *be in* $\mathscr{D}$ and consider the function ψ defined in a neighbourhood of the strip $\{z\in\mathbb{C}\colon 0\le\operatorname{Re}z\le 1\}$ by:

$$\psi(z)=f(\exp(H_0+zH_1))=f(\exp H_0\cdot\exp zH_1).$$

This function is either $-\infty$ everywhere or it is subharmonic. As $i\boldsymbol{a}\subset\boldsymbol{g}$, it depends only on $\operatorname{Re}z$. Then ψ is subharmonic implies $\psi(x)$ $(x\in\mathbb{R})$ is convex (this means in particular that ψ is never $-\infty$). It follows that $\tilde{f}$ is convex. (If $\tilde{f}$ is $-\infty$ at a point, it must be $-\infty$ everywhere).

c) *Surjectivity of $f\mapsto\tilde{f}$.*

Let h be a W-invariant convex function on $\mathscr{D}$. As $\mathscr{D}$ is convex, one has: $h=\sup_{i\in I}g_i$, where the g_i are affine. As h is W-invariant, one also has: $h=\sup_{i\in I}\hat{g}_i$. It suffices to prove that $\hat{g}_i=\tilde{K}_i$ where $K_i\in P_G(D)$. For then, we would have $h=\tilde{K}$ where $K=\sup K_i$ and the function K is plurisubharmonic for the following two reasons: As h is convex, it is continuous and therefore so is K; also K is the supremum of plurisubharmonic functions. So it suffices to prove that for g linear one has $\hat{g}=\tilde{K}$ with K plurisubharmonic. By the lemma, one has

$\hat{g}=\sum_{i=1}^{p} t_i\omega_i+\sum_{j=1}^{r} s_j\mu_t$ with $t_i\geq 0$ $(\forall_i)$. Therefore it suffices to prove the theorem for the ω_i and the $s_j\mu_j$.

Case 1: $(s_j\mu_j)$.

By the definition of the μ_j, there exists a holomorphic character χ_j on $G_{\mathbb{C}}$ such that $\chi_j(\exp H)=e^{\mu_j(H)}$ for $H\in \boldsymbol{a}$. As one is in class 1 $\chi_j(k)=1$ for $k\in K_{\mathbb{C}}$. On the other hand, by compactness of G, we have $|\chi_j(g)|=1$ for $g\in G$. Therefore $(s_j(\log|\chi_j|))^{\sim}=s_j\mu_j$. But the function $s_j\log|\chi_j|$ is plurisubharmonic for χ_j is holomorphic and does not vanish.

Case 2: (ω_i).

Let $\omega=\omega_i$, and let (E_ω,ϱ) be a representation space of class 1, with highest weight ω. We choose on E_ω a G-invariant Hermitian scalar product. Denote by $\|M\|$ the operator norm of $M\in \mathrm{End}_{\mathbb{C}}(E_\omega)$. The function $M\to\log\|M\|$ is plurisubharmonic ([Le1], p. 7–8, prop. [1]). Put $\psi(g)=\frac{1}{2}\log\|\varrho(g\theta(g)^{-1})\|$. One has the following properties:

a) $\psi(g)$ is plurisubharmonic, being a composition of a plurisubharmonic function with a holomorphic map.

b) The map ψ is left G-invariant, as the operators $p(g)$ $(g\in G)$ are unitary. It is also right $K_{\mathbb{C}}$-invariant by the properties of θ.

c) As for $a\in A$ the relation $\theta a=a^{-1}$ is satisfied one has: $\psi(e^H)=\frac{1}{2}\log\|\varrho(e^{2H})\|$ for $H\in\boldsymbol{a}$. As $\varrho(e^{2H})$ is Hermitian, its norm is equal to the maximum modulus of its eigenvalues.

For $H\in\boldsymbol{a}^+$ (the positive Weyl chamber in $\boldsymbol{a}$), this norm is then equal to $e^{2\omega(H)}$. We have $\psi(e^H)=\omega(H)$ for $H\in\boldsymbol{a}^+$, and by W-invariance $\psi(e^H)=\hat{\omega}(H)$, $\forall H\in\boldsymbol{a}$. Therefore $\tilde{\psi}=\hat{\omega}$, where ψ satisfies the hypothesis of the theorem, which finishes the proof for part a of the theorem.

Part *b*) follows from part *a*).

PROOF OF THEOREM 1. *c*) We first note that if $f\in P_G^+(D)$, its restriction to $A_{\mathbb{C}}\cap D$ is also strictly plurisubharmonic, and then it is classical [Hö] that $\tilde{f}$ is strictly convex. By Matsushima's theorem [Ma], the manifold $X_{\mathbb{C}}$ is Stein, and there exists a strictly plurisubharmonic regular exhaustion function h which can be supposed to be G-invariant by integration over G.

Now we must show that if $\tilde{f}\in \mathrm{Conv}_W^+(\mathscr{D})$ then $f\in P_G^+(D)$. We note that for all $x\in\mathscr{D}$, there exists an open convex W-invariant set δ containing x, which is relatively compact in $\mathscr{D}$. It is obtained by taking the W-saturation of a relatively compact neighbourhood of x in $\mathscr{D}$, and then by taking its convex hull. Then there exists $\varepsilon>0$ such that $\tilde{f}-\varepsilon\tilde{h}$ is convex on δ. By *a*) the function $u=f-\varepsilon h$ is plurisubharmonic on the G-invariant domain Δ contained in D with basis δ. Then $f=\varepsilon h+u$ is strictly plurisubharmonic on Δ, and as the point x is arbitrary, one has $f\in P_G^+(D)$.

II APPLICATION TO G-INVARIANT KÄHLERIAN METRICS ON $X_{\mathbb{C}}$

In the following, we take G semi-simple and K connected. As a consequence of our previous results, we obtain a complete classification of G-invariant Kählerian metrics for G-invariant pseudo-convex domains D in the case $H^2(X_{\mathbb{C}}, \mathbb{R})=0$. Some partial results are also discussed in the case $H^2(X_{\mathbb{C}}, \mathbb{R})\neq 0$. As a preliminary remark, using the fact that $\mathscr{D}$ is convex and therefore homeomorphic to A, one obtains from the decomposition $X_{\mathbb{C}}=GAx_0$, $D=G\mathscr{D}x_0$ ($x_0=eK_{\mathbb{C}}$), that $X_{\mathbb{C}}$ is homeomorphic to D. In particular they have the same Betti numbers.

LEMMA 1. *Let M be a manifold and O the sheaf of holomorphic functions. If $H^1(M,O)=H^2(M,\mathbb{R})=0$, then every closed form ω of type $(1,1)$ is given by: $\omega=i\partial\bar\partial\varphi$ where $\varphi\in C^\infty(M)$.*

PROOF. As ω is closed and $H^2(M,\mathbb{R})=0$, we have $\omega=d\alpha$. Write $\omega=\alpha^{1,0}+\alpha^{0,1}$ with $\alpha^{1,0}$ of type $(1,0)$ and $\alpha^{0,1}$ of type $(0,1)$. We have:

$$\omega=d\alpha=(\partial+\bar\partial)(\alpha^{1,0}+\alpha^{0,1})=\partial\alpha^{1,0}+\bar\partial\alpha^{0,1}+\partial\alpha^{0,1}+\bar\partial\alpha^{1,0}.$$

Comparing types we obtain:

$$\omega=\partial\alpha^{0,1}+\bar\partial\alpha^{1,0}\quad\text{and}\quad\partial\alpha^{1,0}=\bar\partial\alpha^{0,1}=0.$$

As $H^1(M,O)=0$ we have $\alpha^{1,0}=\partial\varphi_1$ and $\alpha^{0,1}=\bar\partial\varphi_2$. Therefore $\omega=i\partial\bar\partial\varphi$, where $\varphi=i(\varphi_1-\varphi_2)$.

REMARK. If ω is real, using conjugation, we can arrange φ to be real.

LEMMA. *Let M be a complex manifold with $H^1(M,\mathbb{C})=0$. Then a real function φ on M with $\partial\bar\partial\varphi=0$ is the real part of a holomorphic function.*

PROOF. We have $d(\partial\varphi)=(\bar\partial+\partial)(\partial\varphi)=0$. As $H^1(M,\mathbb{C})=0$ we have $\partial\varphi=du$. This implies $\partial\varphi=\partial u$ and $\bar\partial u=0$, which means that u is holomorphic. Also $\partial(\varphi-u)=0$ so $(\varphi-u)=v$ is antiholomorphic. As φ is real we have $\varphi=\mathrm{Re}(u+v)=\mathrm{Re}(u+\bar v)$ with $u+\bar v$ holomorphic.

The existence of a quasipotential for a closed $(1,1)$-form on $G_{\mathbb{C}}/H$, where $G_{\mathbb{C}}$ is semisimple, was first observed by A.T. Huckleberry.

PROPOSITION 1. A) *Let Ω be a domain in $G_{\mathbb{C}}$ with $H^1(\Omega,O)=H^1(\Omega,\mathbb{C})=H^2(\Omega,\mathbb{C})=0$ and suppose Ω is left G-invariant and right-invariant by a subgroup H of G. Let η be a real closed $(1,1)$-form on Ω with the same invariant properties. Then there exists, up to an additive constant, a unique left G-invariant function φ on Ω and an additive character c of H in $\mathbb{R}$ such that: $i\partial\bar\partial\varphi=\eta$ and $\varphi(gh)=\varphi(g)+c(h)$ $\forall g\in\Omega$, $h\in H$.*

B) *Let L be a complex Lie group and H a closed subgroup of L. Let Ω be a right H-invariant domain in L and φ a right H-invariant function on Ω such* *that*: $\varphi(g\cdot h)=\varphi(g)+c(h)$ *for some character* c *of* H $(g\in\Omega,\ h\in H)$. *Then the form* $\partial\bar\partial\varphi$ *descends to* Ω/H.

PROOF. A) Unicity of φ: Suppose $\partial\bar\partial\varphi=0$ with φ K-invariant on Ω. By lemma 2) one has: $\varphi=\operatorname{Re} f$ with f holomorphic. Let $g\in G$ and $x\in\Omega$. By G-invariance one has: $f(gx)-f(x)\in i\mathbb{R}$. Fixing g and using that f is holomorphic, this gives: $f(gx)-f(x)=d(g)$ (independent of x). Then one sees that d is a character on G. But as G is semisimple, one has $d=0$. As $G_{\mathbb{C}}$ is the complexification of G, this implies that f and then also φ are constant.

Existence

Using Lemma 1, there exists φ such that $i\partial\bar\partial\varphi=\eta$. By integration, one can suppose that φ is left G-invariant. Considering $\varphi(xh)-\varphi(x)$ with $h\in H$ and using the previous part, one has $\varphi(xh)-\varphi(x)=c(h)$ where c is a character on H.

B) As $L\xrightarrow{p}L/H$ is a principal bundle, for every $x\in L/H$ there exists a section s on a neighbourhood U of x. On $p^{-1}(U)$ the function φ can be written as a sum of functions $\tilde\varphi$ and $\tilde c$ defined by: $\tilde\varphi(v\cdot h)=\varphi(v)$ and $\tilde c(v\cdot h)=c(h)$, for $v\in s(U)$ and $h\in U$. As s is biholomorphic, the form $\partial\bar\partial\tilde\varphi$ goes down. To finish the proof, we must see that the function c defined on H is such that $\partial\bar\partial c=0$. Let Z_1, Z_2 be two left H-invariant holomorphic vector fields on H. One has $(\partial\bar\partial c)(Z_1,Z_2)=Z_1\bar Z_2 c$. But as c is a character, $\bar Z_2 c$ is constant, and therefore $Z_1\bar Z_2 c=0$.

REMARKS. 1) This theorem applies when H is complex and closed and $\eta=p^*\omega$, where ω is a $(1,1)$-closed G-invariant form on $p\Omega$. One says that φ is a quasi potential for ω.

2) Suppose in A) that Ω is Stein. Then automatically $H^1(\Omega,O)=0$ ([Hö]). But as Ω is G-invariant, it is also automatic that $H^1(\Omega,\mathbb{C})=H^2(\Omega,\mathbb{C})=0$ as Ω is geodesically convex and then contractible to G ([Lo]).

3) The proposition gives a characterization of $(1,1)$-closed G-invariant forms on Stein domains V of $\mathbb{G}_{\mathbb{C}/H}$ which are G-invariant in terms of quasi-potentials φ on G (H closed and complex). In view of this, one uses Matsushima's theorem [Ma] on principal fibre bundles and then remark 2) for $\Omega=p^{-1}V$.

In the case of Stein G-invariant domains D in $G_{\mathbb{C}}/K_{\mathbb{C}}$ with K semi-simple, one has a complete classification of G-invariant Kählerian metrics. Observe that in this situation, the identity e is in D ([La2]).

THEOREM 2. *If* K *is semi-simple there is a* $1-1$ *correspondence between* G-*invariant Kählerian metrics* ω *on* D *and* W-*invariant strictly convex functions* f *which are zero at* $0\in D$. *The correspondence is as follows*: *one has* $\omega=i\partial\bar\partial g$ G-*invariant and* $g(e)=0$, *take* $f=\tilde g$ (*in the notation of theorem* 1).

PROOF. Using D Stein, one applies the proposition and the remarks to $p^*\omega$. As $K_{\mathbb{C}}$ is semi-simple, one has $c=0$. The result follows from theorem 1, part c.

APPENDIX: WEAKLY G-INVARIANT KÄHLERIAN METRICS ON $G_{\mathbb{C}}/P$ (P PARABOLIC, G-COMPACT, SEMISIMPLE)

DEFINITION. *A weakly G-invariant Kählerian metric on $G_{\mathbb{C}/P}$ (P parabolic) is a real closed $1-1$ form which is positive (not necessarily positive definite).*

A description of such metrics is given in Borel and Hirzebruch [Bo-Hi]. Here we want to describe the quasi potentials of such metrics using the method of part II. Before doing this we have to recall some facts about weights and roots of Lie groups.

Fix a maximal torus T of G and a Borel subgroup B of $G_{\mathbb{C}}$ containing $T_{\mathbb{C}}$. Let R be the roots of $T_{\mathbb{C}}$ in $G_{\mathbb{C}}$, R^+ be the positive system of roots defined by the pair $(B, T_{\mathbb{C}})$ and S the corresponding simple system of roots. One knows that for each $\alpha \in R^+$ there exist $X_\alpha, X_{-\alpha} \in \mathrm{Lie}(G_{\mathbb{C}})$ such that the map $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \mapsto X_\alpha$, $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \mapsto X_{-\alpha}$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \mapsto [X_\alpha, X_{-\alpha}]$ is an isomorphism of $sl(2,\mathbb{C})$ onto the Lie algebra generated by $X_\alpha, X_{-\alpha}$ [Se]. Consequently, there exist homomorphisms φ_α from $SL(2,\mathbb{C})$ onto a subgroup L_α whose Lie algebra is generated by $X_\alpha, X_{-\alpha}$. We set $\alpha^\vee(z) = \varphi_\alpha\left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right)$ and denote the lattice of one parameter subgroups generated by $\alpha^\vee (\alpha \in R)$ by $X^\vee(T)$. Let $X(T)$ be the lattice of characters of T. These lattices are in $\mathbb{Z}$-duality by the pairing $\langle\,,\,\rangle$ defined by $(\chi \circ \alpha^\vee)(z) = z^{\langle \chi, \alpha^\vee \rangle}$ $(\chi \in X(T),\ \alpha^\vee \in X^\vee(T),\ z \in \mathbb{C}^*)$ [Se]. One know that a character χ occurs as the highest weight of an irreducible representation of $G_{\mathbb{C}}$ if and only if $\langle \chi, \alpha^\vee \rangle \geq 0$ $\forall \alpha \in S$. Let $S = \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots. The fundamental dominant weights ω_i are defined by $\langle \omega_i, \hat{\alpha}_j \rangle = \delta_{ij}$ $(i, j = 1, \ldots, n)$. In case G is simply connected, the fundamental dominant weights generate the lattice of characters.

Let P be a parabolic subgroup of $G_{\mathbb{C}}$ containing B and L the maximal reductive subgroup of P containing $T_{\mathbb{C}}$. The group $B \cap L$ is a Borel subgroup of L and the simple system of roots of L defined by the pair $(B \cap L, T_{\mathbb{C}})$ is a subsystem, say, π, of S. We have $P = LR_u(P)$, where $R_u(P)$ is the unipotent radical of P and its Lie algebra is generated by roots vectors X_α with α being the positive roots whose support lies outside π. Also $L = T_1 \cdot L'$, where $T_1 = \{\prod_{\alpha_j \in S \setminus \pi} \alpha_i^\vee(z_i) \colon z_i \in \mathbb{C}^*\}$. Therefore $P = T_1 \cdot L' \cdot R_u(P)$. Since $R_u(P)$ is in the derived group P' of P and every character vanishes on the derived group, we see that any additive character φ of P is of the form

$$(*) \qquad \varphi\Big(\prod_{\alpha \in S \setminus \pi} \alpha^\vee(z_\alpha) \cdot l \cdot r\Big) = \sum_{\alpha \in S \setminus \pi} c_\alpha \log |z_\alpha| \quad (l \in L',\ r \in R_u(P)).$$

For every fundamental weight ω_α, let $(E_\alpha, \varrho_\alpha)$ be the corresponding $G_{\mathbb{C}}$ module with $\|\ \|$ a G-invariant Hermitian norm on E_α and v_α a highest weight vector of unit length. Since $G = KP$ we see using $(*)$ that a K-invariant function ψ on G which restricts to an additive character of P is of the form $\psi(g) = \sum_{\alpha \in S \setminus \pi} c_\alpha \log \|\varrho_\alpha(g) v_\alpha\|$, $c_\alpha \in \mathbb{R}$.

Let ω be a closed $(1,1)$ form on $G_{\mathbb{C}/P}$ and ψ a quasi potential for ω in the sense of proposition 1 A).

THEOREM. *The quasi-potential* $\psi(g)=\sum_{\alpha\in S\setminus\pi} c_\alpha \log\|\varrho_\alpha(g)\cdot v_\alpha\|$ *is associated to a weakly Kählerian form* ω *if and only if* $c_\alpha\geq 0$. *It is associated to a Kählerian metric if and only if* $c_\alpha>0$ $\forall\alpha\in S\setminus\pi$.

PROOF. Let $p: G_{\mathbb{C}}\to G_{\mathbb{C}/P}$ be the canonical projection. For a positive root α let φ_α be a homomorphism from $SL(2,\mathbb{C})$ onto the subgroup L_α of $G_{\mathbb{C}}$ whose Lie algebra is generated by the root vectors $X_\alpha, X_{-\alpha}$. For $\alpha\in S\setminus\pi$ we have $L_\alpha/L_\alpha\cap P\cong\mathbb{P}^1(\mathbb{C})$. Now $p^*\omega=i\partial\bar\partial\psi$ and for $\alpha\in S\setminus\pi$, the restriction of ψ to L_α is $c_\alpha\log\|\varrho_\alpha(g)v_\alpha\|$ $(g\in L_\alpha)$. Therefore if $i\partial\bar\partial\psi$ is a positive form we must have $c_\alpha\geq 0$ $\forall\alpha\in S\setminus\pi$. In case ω is Kählerian we must actually have $c_\alpha>0$ for all $\alpha\in S\setminus\pi$ as in this case $L_\alpha/L_\alpha\cap P$ is a projective line. Conversely if $c_\alpha\geq 0$, then $\psi(g)=\sum_{\alpha\in S\setminus\pi} c_\alpha\log\|\varrho_\alpha(g)\cdot v_\alpha\|$, being a non-negative combination of plurisubharmonic functions is itself plurisubharmonic. Finally if $c_\alpha>0$ $\forall\alpha\in S\setminus\pi$ then $i\partial\bar\partial\psi$ descends to a $(1,1)$-form on $G_{\mathbb{C}/P}$ which is strictly positive on the lines $L_\alpha/L_\alpha\cap P$ $(\alpha\in S\setminus\pi)$. Since the tangent spaces to these lines at $\xi_0=eP$ span the tangent space at ξ_0, and since these tangent lines are orthogonal relative to any T-invariant Hermitian form, we see that $i\partial\bar\partial\psi$ descends to a form which is positive at ξ_0. By G-invariance, it is a positive everywhere on $G_{\mathbb{C}/P}=G/G\cap P$.

ACKNOWLEDGEMENTS

The authors thank the referee for suggesting several improvements and for his most careful reading of the manuscript.

REFERENCES

[Al-Pe] Alekseevskii, D.V. and A.M. Perelomov - Invariant Kählerian metrics on compact homogeneous spaces. Functional Analysis and Applications, 171-183 (1986).

[Bo-Hi] Borel, A. and F. Hirzebruch - Characteristic classes and homogeneous spaces. Amer. J. Math. **80**, 459-538 (1958).

[Co-Lo] Couré, G. and J.J. Loeb - Univalence de certains enveloppes d'holomorphie. C.R. Acad. Sc. Paris **302**, Série I, no. 2, 1986.

[F-J] Flensted-Jensen, M. - Spherical functions on a real semisimple Lie group. A method of reduction to the complex case, J. of funct. analysis **30**, 106-146 (1978).

[He] Helgason, S. - Differential geometry and symmetric spaces. New York, Academic Press, 1978.

[Hö] Hörmander, L. - An introduction to complex analysis in several variables. D. Van Nostrand, New York, 1966.

[La1] Lassalle, M. - Sur la transformée de Fourier-Laurent dans un group analytique complexe réductif. Ann. Inst. Fourier, Vol. **28**, 1978.

[La2] Lassalle, M. - Séries de Laurent des fonctions holomorphes dans la complexifications d'un espace symétrique compact. Ann. Sci. Norm. Sup., 4e série, **11**, 167-210 (1978).

[La3] Lasalle, M. - Une généralization du "Théorème des trois cercles" de Hadamand. Math. Ann. **249**, 17-26 (1980).

Le] Lelong, P. - Séminaire P. Lelong, Lecture Notes no. **474**, Springer-Verlag (1973-74).

[Lo] Loeb, J.J. - Pseudo-convexité des ouverts invariants et convexité géodésique de certains espaces symétriques. Séminaire d'Analyse (P. Lelong, P. Dolbeault, H. Skoda), Lecture Notes, Springer-Verlag (1983-84).

[Ma] Matsushima, Y. - Espaces homogènes de Stein des groupes de Lie complexes. Nagoya Journal **16** (1960).

[Ro] Rothaus, O. - Envelopes of holomorphy of domains in complex Lie groups. Problems in Analysis, Princeton University Press, p. 309-317 (1970).

[Se] Serre, J.P. - Complex semisimple Lie algebras. Springer Verlag, 1987.

[Ta] Takeuchi, M. - Polynomial representations associated with symmetric bounded domains. Osaka J. Math., Vol. **10**, p. 441-475 (1973).